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Reconstruction of Band-Limited Functions from Values on Real Sequences with an Accumulation Point

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A scheme is presented to recover a band-limited function f of finite energy from its sampling values on real sequences with an accumulation point. The result given in this paper can also be viewed as an approach to the extrapolation problem of determinating a band-limited function in terms of its given values on a finite interval. An error estimate is also obtained. © 1997 Academic Press

1. INTRODUCTION

Let $f: T \to C$ be an W-band-limited complex-valued function of finite energy on the real line R, i.e., $f \in L^2(R)$ and $\hat{f}(w) = 0$ outside $[-W, W]$, where $W > 0$ and

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t)\, e^{-itw}\, dt \qquad (w \in R) \tag{1}$$

is the Fourier transform of f. We have

$$f(t) = \frac{1}{2\pi} \int_{-W}^{W} \hat{f}(w)\, e^{iwt}\, dw \qquad (t \in R). \tag{2}$$

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By the Paley–Wiener theorem [2, pp. 103], f can be viewed as the restriction of the entire function of exponential type at most W:

$$f(z)=\frac{1}{2\pi}\int_{-W}^{W}\hat{f}(w)\,e^{iwz}\,dw \qquad (z\in C, \text{ the complex plane}) \tag{3}$$

to the real line R (so a band-limited function is always continuous). Theoretically, the uniqueness theorem for analytic functions implies that f can be wholly determined by its values at any sequence of different (interpolating) points with accumulation points in R. The goal of this paper is to present a scheme to recover a W-band-limited function f of finite energy from its sampling values at a convergent sequence of different points x_n, $n=1, 2, ...,$ with limit $a\in R$.

This can be viewed as an approach to the (irregular) sampling problem of band-limited functions which asks under which conditions and how a band-limited function can be rebuilt if it is known only at a discrete set of points. Because of its great importance in information theory, signal precessing and other application fields, a lot of work has been carried out on (regular and irregular) sampling problems [1, 3, 6, 9, 12 etc.]. We note that sampling theorems in literature demand that the sampling points are dense enough and well scattered for the regular case or "relatively well scattered" for the irregular case on the whole line R. The discrete set of sampling points are equally spaced for the classical Shannon–Whittaker–Kotel'nikov (regular) sampling theorem, and some kind of density (e.g., δ-density used in [6]) for the sampling set is required for irregular sampling problems. Our case is quite different, this kind of requirement is not demanded here. This is one characteristic of the scheme presented in this paper. Because our sampling (interpolating) points x_n, $n=1, 2, \ldots$ converge to a limit a, all but finite sampling points will be within an interval, say $[\alpha, \beta]$. Thus our result can also be viewed as an approach to the extrapolation problem of determinating a band-limited function in terms of its given values on an interval $[\alpha, \beta]\subset R$ [10]. So another characteristic of the recovery scheme presented here is that it can be used to rebuild the function from its value on any nonempty interval. As the referees pointed out, this scheme has drawbacks, to calculate the first n coefficients of the series, one needs to solve n linear equations in n unknowns and the stability can not be ensured. From this point of view, it would be proper to view our scheme more as an extrapolation method than as a sampling method.

2. MAIN RESULT

Suppose $\{x_n, n=1, 2, ...\}\subset R$, $x_j\neq x_k$ $(j\neq k)$, and $x_n\to a (n\to\infty)$. If $\{f(x_n), n=1, 2, ...\}$ are known, we hope to reconstruct f from $\{f(x_n), n=1, 2, ...\}$.

Since we can consider $f_a(x) = f(x+a)$ with the sampling points $\{x_n - a, n = 1, 2, ...\}$, we may assume $a = 0$ without loss of generality. Furthermore, we may assume $\{x_n\}$ to be monotone decreasing because otherwise we can consider a monotone decreasing convergent subsequence of $\{x_n\}$ or a monotone decreasing convergent subsequence of $\{-x_n\}$ and $f^-(x) = f(-x)$.

THEOREM 1. *Let $f: R \to C$ be a W-band-limited function of finite energy, $\{x_n, n = 1, 2, ...\}$ be a monotone decreasing sequence and $x_n \to 0 (n \to \infty)$, then*

$$\phi_m(t) = \sum_1^m E_m(k)\, e^{ikWt/m} e^{-iWt} \tag{4}$$

converges to $f(t)$ uniformly on each compact subset $S \subset R$ when $m \to \infty$, where the coefficients $E_m(k)$ are chosen so that the interpolation equations

$$f(x_n) = \phi_m(x_n) \qquad (1 \leqslant n \leqslant m) \tag{5}$$

are satisfied, i.e., $\{E_m(k), k = 1, ..., m\}$ is the solution of the system of linear equations

$$f(x_n) = \sum_1^m E_m(k)\, e^{ikWx_n/m} e^{-iWx_n} \qquad (1 \leqslant n \leqslant m). \tag{6}$$

Proof. For simplicity, we only prove the theorem for $x_n = 1/n$ $(n = 1, 2, ...)$. The general case can be proven similarly.

Under the above assumption, (6) becomes

$$f(1/n) = \sum_1^m E_m(k)\, e^{ikW/(mn)} e^{-iW/n} \qquad (1 \leqslant n \leqslant m). \tag{7}$$

It is easy to verify that the determinant of coefficients for (7) is nonsingular, so the system of linear equations (7) has a unique solution $\{E_m(k), k = 1, ..., m\}$.

Take real numbers r_1 and r_2 such that

$$0 < r_1 < r_2 < 1 \qquad \text{and} \qquad 0 < \frac{2r_1(1+r_2)^2}{r_2 - r_1} < 1. \tag{8}$$

For each $m \geqslant 1$, the function

$$g_m(z) = z^m \cdot \frac{1}{2\pi} \int_{-W}^{W} \hat{f}(w)\, z^{mw/W}\, dw \qquad (z \in C) \tag{9}$$

and the polynomial

$$p_m(z) = \sum_1^m E_m(k)\, z^k \qquad (z \in C) \tag{10}$$

are analytic on $D_2 = \{z\colon |z-1| < r_2\} \supset D_1 = \{z\colon |z-1| < r_1\}$.

Take a positive integer M such that the complex numbers

$$z_{m,n} = e^{iW/(mn)} \qquad (1 \leqslant n \leqslant m) \tag{11}$$

are in D_1 when $m > M$. It follows from (9), (2), (7) and (10) that

$$\begin{aligned} g_m(z_{m,n}) &= (z_{m,n})^m \cdot \frac{1}{2\pi} \int_{-W}^{W} \hat{f}(w)(z_{m,n})^{mw/W}\, dw \\ &= e^{iW/n} \cdot \frac{1}{2\pi} \int_{-W}^{W} \hat{f}(w)\, e^{iw/n}\, dw \\ &= e^{iW/n} f(1/n) \\ &= e^{iW/n} \sum_{k=1}^{m} E_m(k)\, e^{ikW/(mn)} e^{-iW/n} \\ &= p_m(z_{m,n}). \end{aligned} \tag{12}$$

Thus $p_m(z)$ is the interpolating polynomial of degree at most m with the values $\{g_m(z_{m,n}), n = 1, ..., m\}$ at the points $\{z_{m,n}, n = 1, ..., m\}$. By the Hermite theorem [4, p. 68], we have for $m > M$ and $z \in D_1$

$$g_m(z) - p_m(z) = \frac{1}{2\pi i} \int_{C_2} \frac{(z - z_{m,1})(z - z_{m,2}) \cdots (z - z_{m,m})\, g_m(t)}{(t - z_{m,1})(t - z_{m,2}) \cdots (t - z_{m,m})(t - z)}\, dt, \tag{13}$$

where $C_2 = \{z\colon |z-1| = r_2\}$ is the boundary of D_2. It follows directly from (9) that

$$\begin{aligned} |g_m(t)| &\leqslant \frac{(1+r_2)^m}{2\pi} \int_{-W}^{W} |\hat{f}(w)|\, (1+r_2)^{m|w|/W}\, dw \\ &\leqslant (1+r_2)^{2m} \frac{1}{2\pi} \int_{-W}^{W} |\hat{f}(w)|\, dw \\ &= A(1+r_2)^{2m} \qquad (m > M,\ t \in C_2) \end{aligned} \tag{14}$$

with the constant

$$A = \frac{1}{2\pi} \int_{-W}^{W} |\hat{f}(w)| \, dw \leqslant \frac{1}{2\pi} \left(\int_{-W}^{W} |\hat{f}(w)^2| \, dw \right)^{1/2} \cdot (2W)^{1/2} \leqslant \infty \quad (15)$$

based on the Cauchy–Schwarz inequality and the fact that f is of finite energy. Furthermore, we can verify for $m > M$, $1 \leqslant n \leqslant m$, $t \in C_2$ and $z \in D_1$ that

$$|z - z_{m,n}| \leqslant 2r_1, \qquad |t - z_{m,n}| \geqslant r_2 - r_1, \qquad |t - z| \geqslant r_2 - r_1. \quad (16)$$

Combining (14), (15), (16) and (13), we obtain

$$|g_m(z) - p_m(z)| \leqslant \left(\frac{2Ar_2}{r_2 - r_1} \right) \left(\frac{2r_1(1 + r_2)^2}{(r_2 - r_1)} \right)^m. \quad (17)$$

Thus it follows from (8) and (17) that

$$\lim_{m \to \infty} |g_m(z) - p_m(z)| = 0 \quad (18)$$

uniformly for $z \in D_1$.

For any fixed compact subset $S \subset R$, take a positive number T such that $S \subset [-T, T]$. It is not difficult to find a positive integer $N > M$ such that $|e^{iWt/m} - 1| < r_1$ for all $m > N$ and $t \in [-T, T]$, this means

$$z(m, t) = e^{iWt/m} \in d_1 \quad (19)$$

for all $m > N$ and $t \in [-T, T]$. Thus it follows from (18) and (19) that

$$\lim_{m \to \infty} |g_m(z(m, t)) - p_m(z(m, t))| = 0 \quad (20)$$

uniformly for $t \in S$. But from (9) and (2), we have

$$g_m(z(m, t)) = e^{iWt} \cdot \frac{1}{2\pi} \int_{-W}^{W} \hat{f}(w) \, e^{iwt} \, dw = e^{iWt} f(t), \quad (21)$$

and from (10) and (4), we have

$$p_m(z(m, t)) = \sum_{1}^{m} E_m(k) \, e^{ikWt/m} = e^{iWt} \phi_m(t). \quad (22)$$

Thus (20), (21) and (22) tell us that

$$\lim_{m \to \infty} |\phi_m(t) - f(t)| = 0 \quad (23)$$

uniformly for $t \in S \subset [-T, T]$.

3. ERROR ESTIMATE

We deal with the special case for $\{x_n\} = \{1/n\}$, because we can handle the general case similarly.

THEOREM 2. *The assumptions are as in Theorem* 1 *with* $1/n$ *replacing* x_n $(n = 1, 2, ...)$. *For any* $0 < \lambda < 1$ *and* $T > 0$, *we can find a positive integer* N *such that*, *for any* $m > N$, *we have the error estimate*:

$$|\phi_m(t) - f(t)| < \frac{22\sqrt{\pi}}{9\pi} \sqrt{WE}\, \lambda^m \qquad (t \in [-T, T]), \tag{24}$$

where $[x]$ *denotes the greatest integer which is less than* x, E *is the energy of* f, *i.e.*,

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-W}^{W} |\hat{f}(w)|^2 \, dw. \tag{25}$$

Proof. Take $r_1 = \lambda/(9 + 2\lambda)$ and $r_2 = 1/2$, then

$$0 < r_1 < r_2 < 1 \qquad \text{and} \qquad \frac{2r_1(1 + r_2)^2}{r_2 - r_1} = \lambda. \tag{26}$$

It follows from (21), (22), (17), (26) and (15) that there is a positive integer N such that, for any $m > N$, and $t \in [-T, T]$,

$$\begin{aligned} |\phi_m(t) - f(t)| &\leqslant \frac{2r_2 A}{r_2 - r_1} \lambda^m \\ &\leqslant \frac{2(9 + 2\lambda)}{9} \sqrt{\frac{WE}{\pi}} \lambda^m \\ &< \frac{22\sqrt{\pi}}{9\pi} \sqrt{WE}\, \lambda^m. \end{aligned} \tag{27}$$

Remark. An interesting topic is pointed out by one of the referees. Although the recovery schemes derived from sampling theory are closely related to complex function theory, the comparison between the classical complex function theory and the sampling schemes deserves some discussion. Surprisingly, very few references of this kind can be found in literature though many authors pointed out that the sampling theory roots deeply in classical complex function theory. The following statement can be found in [9], "Cardinal series have found favor in signal-processing applications, undoubtedly because of the neat way in which it fits into the accompanying Fourier analysis."

Although the sampling theory has the complex function theory as its solid background, it was originally introduced ([7, 8, 11]) from application domain. The sampling recovery schemes are more application-oriented while the methods which come directly from the classical complex function theory are more strict. A fact is that the two approaches are getting closer and no clear boundary exists nowadays.

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